

THE SIZE AND SHAPE OF DANTE'S MOUNT PURGATORY

Paola Magnaghi-Delfino and Tullia Norando

Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano (MI), Italy.

Abstract: Where is Mount Purgatory? How high is it? How large is the island upon which it was situated? In the nineteenth century Rodolfo Benini and Ideale Capasso developed a series of hypotheses and calculations to find answers to these questions. Each used data derived from mathematics, astronomy, history of science and cartography, but they completely disagreed on the location and on the overall size and shape of the island. In this paper we review the main points of these two scholars, then we rework the calculations and estimates, according to a new astronomical hypothesis presented by Giulio Magli and Claudio Facciolo.

1 INTRODUCTION

The authors of this paper are engaged in interdisciplinary research at the FDS Laboratory of the Politecnico di Milano, and, in particular, they are interested in contributing to projects that combine scientific data and artistic insights. The FDS Laboratory operates in mathematical training for teachers, science communication, education, experimental teaching and cultural heritage. These activities are developed in cooperation with universities and secondary schools. Some projects are developed in collaboration with astronomers at the National Institute for Astrophysics (INAF) in Rome.

This research paper is the second in an interdisciplinary research project, which began in 2013 with the publication of the paper "Galileo Galilei's location, shape and size of Dante's Inferno" (Angelini, Magnaghi-Delfino and Norando, 2014). Galileo (1564-1642) was called by the *Accademia della Crusca* to resolve the debated question about the shape and size of *Inferno*, opposing the views proposed by Antonio di Tuccio Manetti (1423-1497) and Alessandro Vellutello (b. 1473). In these lectures, Galileo (1588) combines a clear exposition of mathematics with his deep knowledge of Dante's *Divina Commedia* (*Divine Comedy*) and emphasizes that the geometry of Manetti's plan is based on evidence from the poem. Both plans are based on Dante's geographical knowledge and religious beliefs; the two arguments are identical as regards the general appearance of the *Inferno*, but are considerably different regarding the shape and the size. The learned men of the Renaissance based their mathematical knowledge on such ancient classical texts as the works of Euclid (367-283 BC) and Archimedes (287-212 BC). The Renaissance artists and scientists were deeply rooted in the mathematical and philosophical heritage of Classical Greece and the boundaries between scholarship, art, science and mathematics were inextricably intertwined.

Dante (1265-1321) may be considered the 'founding father' of Italian poetry and he stamped the mark of his lofty and commanding personality upon later literature. It can even be claimed that his works have played a direct role in shaping the aspirations and destinies of his native country. During the nineteenth century, Dante was the subject, at a European level, of a deep and heartfelt rediscovery, both literary and biographical. In the Risorgimento¹, in particular, Dante becomes the symbolic reference of the aspirations and identity of the Nation, of which the Florentine poet is considered the ideal unifier, both linguistically and politically. In the twentieth century in Italy many authors wrote about the content and literary style of the *Divine Comedy*. The poem has achieved great fame, and many scholars study the *Divine Comedy* and try to explain and interpret its scientific content.

In this paper we examine the location, the height and the topography of *Mount Purgatory*. We consider two different points of view: the 1917 study of Rodolfo Benini (1862-1956) and the 1967 book of Ideale Capasso (1905-1992). Then we propose another calculation of the location, height and topography utilizing material in the 2010 thesis of Claudio Facciolo (b. 1962).

The notion expounded by Dante is that Earthly 'Paradise' was situated in the Southern Hemisphere at the summit of Mount Purgatory. The idea of its great altitude, above the reach of all atmospheric disturbances, so as to be fit for the secure abode of man, is commonly found in the works of other writers.

Benini and Capasso differ in their estimates of the size and the shape of the Mount. To Capasso, Mount Purgatory is ~11 km high, whereas Benini favours ~192 km high, much higher than medieval writers would accept; in fact, the maximum height of the Earth's atmosphere in Alfraganus' *Elements of Astronomy* (833) is 110 km, while Alhazen's figure (1972) is 79 km.

Capasso's shape is a truncated cone, the average slope of which did not exceed 30°, and the diameter of the base was about 3.5 times the height. These figures were derived from the verses of Dante.

In contrast, Benini's shape is similar to the structure of a ziggurat. Two examples, based on his concept, are shown in Figure 1.

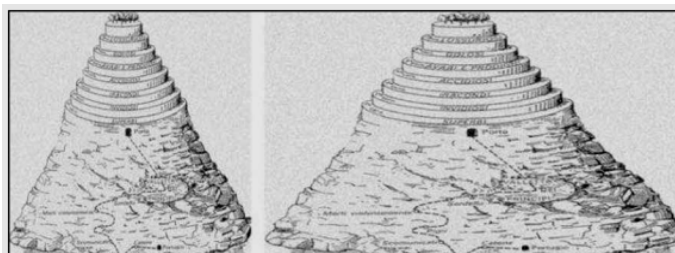


Figure 1: Two different versions of Mount Purgatory according to Benini (1917).

In this research paper we review the main points of each work, then we rework the calculations using an

alternative point of view, involving the position of the Southern Cross. We find a height for Mount Purgatory that is comparable to that of Capasso and a shape that is similar to that suggested by Benini.

Among surviving ziggurats, the inclination of the Aztec Pyramid of the Sun in Teotihuacan (Mexico) is greater than that of Mount Purgatory, as proposed by Capasso. The Mexican pyramid has a square base of 223.5 meters on each side, is 71.2 meters high and has an inclination of about 32° (Reynolds, 1999). We consider the following verses in which Dante quotes some mountains that he knew of:

One can walk to Sanleo and descend to Noli,
one can mount Bismantova to the summit,
with feet alone; but there a man must fly.
(*Purgatorio*, IV, 25-27; Alighieri, 2005; 1977).²

'Bismantova' mentioned in this quotation is a geological formation in the Apeninne Mountains. It has the shape of a narrow plateau measuring 1 km x 240 m, whose steep walls rise ~300 m above the nearby hills. Moreover, in the Santa Maria del Fiore in Florence, Raphael Sanzio (1483-1520) painted a famous picture in which Mount Purgatory has the exact shape of a ziggurat.

Because there are no *Divine Comedy* manuscripts or records provided by Dante's contemporaries, none of these hypotheses can be considered as false. Each author highlights an aspect of Dante's scientific culture, which is often not sufficiently valued by scholars.

2 DANTE'S KNOWLEDGE

Because we do not have documentary data about Dante's culture, we have to reconstruct this from his works, *Divine Comedy*, *Convivio* (*The Banquet*) and *Questio de Aqua et Terra* (*The Question of the Water and the Land*). From these we derive an image of a scholar of theology, philosophy, physics, astronomy, grammar and rhetoric, which were all the disciplines in the *trivium* and the *quadrivium* at medieval schools and universities.

It is likely that Dante studied religious and secular topics in Florence, and we suppose that he took courses at the University of Bologna in 1287 when Guido Bonatti, the author of *Decem Continentes Tractatus Astronomiae* (*Book of Astronomy*, 1851), taught in that city (*Inferno* XX, 118). Dante and Bonatti both learnt astronomy from the book *Elementa Astronomica* (*Elements of Astronomy*) by the ninth century astronomer Ahmad ibn Muhammad ibn Kathir al-Farghàni, also known as Alfraganus. This book is a compendium of material from the *Almagest*, the great work of Ptolemy (who Dante places in Limbo). This work first was translated from Arabic into Latin in the twelfth century by Gerardo da Cremona (1114-1187), who also was the first to translate the *Almagest* into Latin; and again, a little later, by Johannes Hispaniensis of Seville, and would have been accessible to Dante. Alfraganus is quoted in Dante's *The Banquet* (II, xiv. 95) as his authority for the dimensions of the planet Mercury. Again, Dante quotes Alfraganus' work in *The Banquet* (II, vi. 134), under the title *The Book on the Clustering of Stars*. Most of the astronomical data, and sometimes even the comparisons and illustrations, given by Dante are found in Alfraganus, e.g. the graphic comparison of the revolution of the Sun as seen from the poles at the equinox (see *The Banquet*, III, v. 147) like that of a millstone. In Alfraganus' treatise (833) there are four chapters entirely devoted to geography, and a visual representation of the world that Brunetto Latini (1220-1294; n.d.) copies in his maps.

In order to discuss the location of Mount Sinai in ancient cartography we quote the work *Secreta* (or *Liber Secretorum*) *Fidelium Crucis* (*The Book of the Secrets of the Faithful of the Cross*) by Marin Sanudo the Elder (1270-1343; 1972), a Venetian statesman and geographer. This work occupies an important place in the development of cartography. It was begun in March 1306 and finished (in its earliest form) in January 1307, when it was offered to Pope Clement V as a manual for true Crusaders who desired to regain the Holy Land.

Colored maps accompany 9 of the surviving 19 complete texts, which are presumably presentation copies. They were designed specifically with the subject matter of *The Book of the Secrets* ... in mind, and Sanudo made a contribution in selecting the illustrative material, even if he did not draw the maps himself.

Kretschmer (1891; 1909) showed that the fourteenth century Genoese cartographer Pietro Vesconte, who was active between 1310 and 1330, was the person who drew Sanudo's maps. Sanudo adapted the map of Palestine from a Florentine original, and produced the associated town plans of Jerusalem from Burchard of Mount Sion and the plan of Acre from his own memory of his visits in 1280. Conder (1881) noted that the map of Palestine is a rude and very inaccurate sketch, and is not to scale.

The official language of the medieval universities was Latin, hence Dante's humanistic studies were based primarily on Latin authors, and Virgil, in particular, had a decisive influence on Dante's works. Dante learned the tradition of minstrels, the Provençal poets, and he had a particular devotion to Virgil:

You are my master and my author.
You alone are he from whom I took
the fair style that has done me honor.
(*Inferno*, 1,85-87; Alighieri, 2005; 1977).

So we can assume that at this time Dante was an up-to-date intellectual with worldwide interests.

3 THE LOCATION, HEIGHT AND TOPOGRAPHY OF MOUNT PURGATORY ACCORDING TO IDEALE CAPASSO

Mount Purgatory is at the antipodes of Jerusalem, which is at a latitude of 32° N so Mount Purgatory must be 32° S. According to modern maps, Jerusalem is at a longitude of 35° E, so Mount Purgatory would be 145°W, in the Pacific Ocean to the south of the archipelago of Tubuai (which has co-ordinates of 23° 27' S and 149° 30' W).

Dante imagined Mount Purgatory was the highest of all mountains, as can be seen from the following verses. In the description of Ulysses' last voyage he writes:

... when there appeared to us a mountain
dark in the distance, and to me it
seemed the highest I had ever seen.
(*Inferno*, XXVI, 133-135; Alighieri, 2005; 1977).

Here we have Dante's assertion:

... and I turned my face to the hill
that rises highest heavenward from the sea.
(*Purgatory*, III, 14-15; Alighieri, 2005; 1977).

Finally, there is Adam's description:

... on the mountain which rises highest from the sea.
(*Paradise*, XXVI, 139; Alighieri, 2005; 1977).

The necessary information to determine the height of Mount Purgatory which is contained in the XXI, XXVII and XXVIII Cantos of *Purgatory*, is all astronomical and provides discordant results. In the XXI Canto, the Roman poet Publius Papinius Statius (AD 40-96) says that the gate of Purgatory is located at the very limit of the region of clouds and weather disturbances, above the three steps where the Guardian Angel rests. Also, *Matelda* (Canto XXVIII) says that Mount Purgatory is so high that earth and water cannot reach the point where the gate of Purgatory is located.

In order to find references to the height of that atmospheric region one can read the *Liber de Crepusculis (On Twilight)* (Alhazen, 1972; 2007) by al Hasan b. Hayman (965-1039), mentioned by Roger Bacon (1214-1294) in his *Opus Majus (Major Work)* (1962), and in the *Libro II (Book II)* (cap.23) of *Naturalis Historia (Natural History)* by Plinius (AD 23-79; 77).

In *On Twilight*, Alhazen demonstrates mathematically that the Earth's atmosphere reaches a maximum height of 51 2/3 miles (where 1 mile = 1,764 meters)³, on the basis of the following data: the Earth's circumference = 20,400 miles; the Sun's diameter = 5 1/2 times the Earth's diameter; the Sun's distance from the Earth = 1,100 Earth radii; and the altitude of the Sun when dusk starts = 18°. In his *Meteora (Meteorology)*, Aristotle (384-322 BC; 1931) says that the atmosphere extends to the limit between the air and fire, so (1962) suggests that we accept a figure of at least 51 2/3 miles for the height (i.e. ~90 km).

In his *Natural History* Plinius (77) accepts the statement by Posidonius of Apamea (135-50 BC; n.d.), who estimated that the maximum height of the clouds was 40 stades (where 1 stade = 185 meters, i.e. about 7.4 km). It must be said that Posidonius also believed that the Earth's circumference was only 16,100 miles.

In Dante's *Canto XXVII* we find useful quotes to determine the height of Mount Purgatory which are based on the difference in time between sunset at the base of the Mount and sunset at a certain height. The Sun sets when the Angel of Chastity appears to Dante. In pinpointing this moment, Dante gives the hours of the four simultaneous key meridians of this geographic system:

As when it darts forth its first beams
there where its Maker sheds His blood,
while Ebro falls beneath the lofty Scales
and the waves in the Ganges are scorched by noon,
so stood the Sun, so that the day was departing
when the glad angel of God appeared to us.
(*Purgatory*, XXVII, 1-6; Alighieri, 2005; 1977).

As the Sun was in the sign of Aries and Libra was opposite, near the Ganges, where it was noon, Aries and the Sun were located together along the meridian. At the same time, at Ebro (in Spain), when Libra crossed the meridian it had to be midnight. So sunset at Mount Purgatory coincided with the rising of the Sun in Jerusalem, with noon to the Ganges with midnight in Spain.

At Mount Purgatory, Dante and Virgil began to climb the stairs that led to the Garden of Eden, but after a few steps they stopped, because the Sun had just set and it quickly became dark. Capasso (1967) estimates that the elapsed time from the start of the ascent to sunset was 15 minutes. Knowing the difference between the time of sunset at the base of Mount Purgatory and sunset at the height reached by Dante, one can find the height of the mountain. In the time interval ΔT between the two sunsets,⁴ the Sun is lower than the astronomical horizon at this place relative to the marine horizon by an angle equal to the apparent depression (AP). This angle is given in minutes by the formula

$$AP = 15 \times \Delta T \times \sqrt{(\cos^2 \varphi - \sin^2 \delta)} \quad (1)$$

where φ is the latitude and δ the declination of the Sun. For $\Delta T = 15$ minutes and so $15 \times \Delta T$ arcmin, $\varphi = 32^\circ$, $\delta = 6^\circ$, we have $AP = 189'$. Capasso (*ibid.*) assumes $\delta = 6^\circ$ because he agrees with the calculations of Cantelli (1917) about the date of the arrival of Dante and Virgil at Purgatory. Cantelli asserts that the poets came to Purgatory on the night before 27 March 1301 (cf. Angelitti, 1897). On this day the declination of the Sun was $+ 6^\circ$. Since the elevation, e , is approximately

$$e = (AP/1.8)^2 \quad (2)$$

we get $e = 11,000$ meters. Since the entrance to the Garden of Eden (on the top of Mount Purgatory) is nearby, the mountain should not be much higher.

From the height of the mountain we can obtain the diameter of the base, knowing the inclination of the slope,

which can be deduced from Dante's poetry. At the beginning, the climb is very steep and strenuous, greater than 45°, but not for very long because it only takes Dante and Virgil about 2 hours:

We came meanwhile to the foot of the mountain.
 Here we found the cliff so steep that in vain would
 legs be nimble there ...
 - Now we know on which side the hillside slopes
 said my master, stopping,
 -so that he can ascend who goes without wings?
 (*Purgatory*, III,46-48; 52-54; Alighieri, 2005; 1977).

So high was the top that it surpassed my sight,
 and the slope was far steeper than a line
 from mid-quadrant to the center.
 (*Purgatory*, IV, 40-42; Alighieri, 2005; 1977).

Then the slope becomes gentler, and on this part of the coast we can deduce the inclination from the instant when the Sun is hidden behind it. Placing such an instant between 3 p.m. and 4 p.m., we obtain an inclination of between 37° and 23°:

And he [said] to me: "This mountain is such, that ever
 at the beginning down below it is toilsome,
 but the higher one goes the less it wearies [you].
 (*Purgatory*, IV, 88-90; Alighieri, 2005; 1977).

In Dante's *Purgatory* the slope is not exceptional:

From its edge, bordering the void,
 to the foot of the high bank which rises sheer,
 a human body would measure in three lengths⁵
 (*Purgatory*, X, 22-24; Alighieri, 2005; 1977).

and said "Come: the steps are at hand here,
 and henceforth the climb is easy."
 (*Purgatory*, XII, 92-93; Alighieri, 2005; 1977).

The slope for the hot-tempered is easier and even less inclined:

With a glad voice he said, "Enter here
 to a stairway far less steep than the others."
 (*Purgatory*, XV, 35-36; Alighieri, 2005; 1977).

The climb for the extremely greedy is easy:

And I, lighter than at the other passages,
 went on so that without any toil
 I was following the fleet spirits upwards ...
 (*Purgatory*, XXII, 7-9; Alighieri, 2005; 1977).

Then the way that finally leads to the Garden of Eden is almost flat:

In order that the disturbance which
 the exhalations of the water and of the hearth
 (which follow so far as they can the heat)
 produce below might do no hurt [harm] to man,
 this mountain rose thus high towards heaven,
 and stands clear of them from where it is locked ...
 such movement strikes upon this height,
 which is wholly free in the pure air⁶
 (*Purgatory*, XXVIII, 97-102,106-107; Alighieri, 2005; 1977).

If the average slope does not exceed 30° then the diameter of the base will be about 3.5 times the height (see Table 1). For a comparison of the size of Mount Purgatory island and other known islands see Table 2.

Table 1: Mount Purgatory's measurements according to Capasso (1967).

Height	$h = 11 \text{ Km}$
Average slope	30°
Base diameter	$d = 38.5 \text{ Km}$
Circumference of the base	$C = \pi d = 121 \text{ Km}$
Island Area	$A = \pi d^2/4 = 1164 \text{ Km}^2$
Volume of Purgatory	$V = A h/3 = 4269 \text{ Km}^3$

Table 2: Purgatory Island and the sizes of some other known islands.

Island	Size (km ²)
Purgatory Island	1164
Rhodes	1398
Aracena (Chile)	1164
Grande Comore	1158
Martinique	1101
Elba	223
Ithaca	96

4 THE LOCATION OF MOUNT PURGATORY ACCORDING TO RODOLFO BENINI

Mount Purgatory is located on the Earth exactly opposite Zion (its antipodal point), and Virgil says to Dante:

... imagine Zion
 and this mountain to be so placed on the Earth
 that they have one sole horizon
 and different hemispheres; then you will see
 that the way which Phaethon,
 unhappy for him, knew not how to drive,
 must need to pass this mountain on the one side
 when it passes that mountain on the other.
 (*Purgatory*, IV, 68-74; Alighieri, 2005; 1977).

The myth of Phaethon was created by Publius Ovidius Naso (43 BC-AD 18) in *Metamorphoses* (*Metamorphosis*), and his term " ... road of the Sun ... " (Ovid, n.d.) can be interpreted as the ecliptic rather than a diurnal arc. Also, in another place, Dante (*Paradise*, X, 16) mentions the oblique circle that includes the Sun and the planets (the ecliptic), calling it " ... their pathway." In this case, we can find Mount Zion on the Tropic of Cancer and Purgatory on the Tropic of Capricorn, and so Mount Purgatory must be at a latitude of about 23°30' S.

The subsequent response of Dante reinforces this interpretation:

That the mid-circle of celestial motion,
 which is called the Equator ...
 ... departs here
 towards the north, as far as the Hebrews
 used to see it toward the hot climes.
 (*Purgatory*, IV, 79-80, 82-84; Alighieri, 2005; 1977).

Dante mentions the place where the Hebrews lived, not the Jews, and therefore in Benini's opinion he does not refer to Jerusalem, but to the Jewish people who lived in Egypt or wandered in the Sinai. According to the uncertain medieval geographers, the size of land masses to the south of the Sinai Peninsula was exaggerated, as we can see in the paper by Marin Sanudo the Elder (1270-1343), where the distance between the extreme south of the Sinai Peninsula and the angle southeast of the Mediterranean equals or exceeds all that section between the southeast corner and the northeast of the Mediterranean, while in reality it is less than half.

In conclusion, Benini (1917) claims that Mount Purgatory was at a latitude of about 23°30' S.

Benini's hypothesis can be confirmed in the following places in the *Divine Comedy*:

- the disappearance of Capricorn
- the setting of Ursa Major
- four circles and three crosses
- the navigation of Ulysses
- the duration of Ulysses' final the voyage.

Let us now examine each of these.

5 ARGUMENTS IN SUPPORT OF RODOLFO BENINI'S HYPOTHESIS

5.1 The Disappearance of Capricorn

In *Purgatory* (II, 55-57; Alighieri, 2005; 1977) Dante writes:

The Sun was shooting forth the day on all sides
 and with his deft arrows had chased Capricorn
 from mid-heaven.

If the expression " ... from mid-heaven." is meant in the literal sense, then the constellation of Capricorn was in the middle of the sky when Dante was at the Tropic. However, according to Capasso (1967), the poet simply said that the first light of dawn obscured the stars in the sky, including those of Capricorn, which were close to the meridian.

5.2 The Setting of Ursa Major

In *Purgatory* (I, 28-31; Alighieri, 2005; 1977), Dante writes:

When I had withdrawn my gate from them, turning a little to the other pole,

there whence the Wain⁷ had already disappeared,
I saw close to me an old man alone.

From the latitude of the Tropic of Capricorn we can see, for varying periods, all the stars in the northern sky from the Pole to a declination of more than 23° 30'. Now in AD 1300, the seven stars of Ursa Major occupied a stretch of sky between 24° 30' and 37° so from the island of Purgatory, in the season when Dante made the journey, at a certain hour of the night they would all have been visible at the same time.

5.3 Four Circles and Three Crosses

After arriving at the top of Mount Purgatory, while Dante is already ascending into heaven of the Moon, he notices that the Sun shines from the first point of Aries, "Which joins four circles with three crosses." (*Paradise*, I, 39; Alighieri, 2005; 1977).

In the opinion of many scholars the four circles are the horizon, the celestial equator, the ecliptic the equinoctial coluro,⁸ and the three crosses are formed by the intersection of the three circles of the ecliptic, the equator and the equinoctial colure, with the circle of the horizon. Capasso (1967) considers this interpretation to be unsatisfactory because none of the angles is a right angle, and he proposes four other circles: the celestial equator, the horizon, the first vertical⁹ and the first hour circle.

The interpretation proposed by Benini (1917) and other scholars is to replace the horizon with the circle of latitude passing through the equinoctial points; now we have two crosses at right angles: the cross formed by the equator and the equinoctial colure and the cross formed by the ecliptic with the circle of latitude. The third cross is the angular space between the celestial equator and the ecliptic and between the equinoctial colure and the circle of latitude (Figure 2).

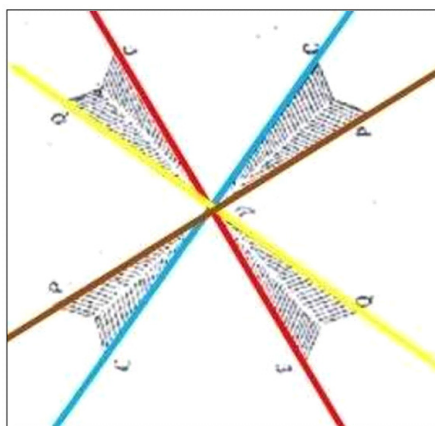


Figure 2: The three crosses. Yellow line = Equator; Red line = Ecliptic; Blue line = Colure; Brown line = Circle of Latitude.

If Mount Purgatory is on the Tropic of Capricorn, that is on the ecliptic, its horizon is the circle of latitude. Then the arms of the third cross form angles of equal amplitude, that is angles of 23° 30'. If Mount Purgatory is placed at the antipodes of Jerusalem, its horizon is not the circle of latitude and so we cannot get three crosses that are symmetrical in appearance (Cantelli, 1917).

5.4 The Navigation of Ulysses

Ulysses steers the ship "... di retro al sol ..." which means following the Sun, but not during the daily motion, but rather the annual motion from Cancer to Capricorn. In fact, when he arrives at the narrow passage where Hercules set his landmarks, Ulysses turns the stern towards the morning, and then he crosses the equinoctial circle and sails on until he only sees stars of the southern circumpolar sky.

Ulysses could not have passed the Tropic, otherwise he would have sailed beyond the Sun, instead of following the Sun. But if we judge that Mount Purgatory had a latitude of 32° S, we can assume that the voyage did extend beyond the Tropic. In fact, the ship would have to be at a latitude of 27 or 28° S for Ulysses to see the top of a mountain that was at least 10 miles high.

5.5 The Length of Ulysses' Final Voyage

The duration of the voyage was 'cinque lune' (or 147 days). Ulysses sights Mount Purgatory, and shortly afterwards crosses the Equator:

The night now saw the other pole
and all its stars, and ours so low that
it did not rise from the ocean floor.
(*Inferno*, XXVI, 127-129; Alighieri, 2005; 1977).

In the fourteenth century, Ursa Major was in the sky between latitudes 24° 30' and 37° S, so Ulysses stopped 7-8° south of the Equator (and 15° north of Mount Purgatory). The sailors had come a distance of 5,381 miles (8,660 km), if they had travelled about 36.6 miles/day (or 58.0 km/day) at an average speed of 3 knots.

This might seem an excessive duration, given that it took the Greeks and Phoenicians five days to voyage from Crete to the mouth of the Nile, travelling at almost double the speed of Ulysses' vessel (Ferrara, 1998; Janni, 1996). But we must remember that these trips were accomplished by rowing day and night, with numerous, strong young oarsmen, whereas Ulysses' companions only helped during the day and were "... old and slow." (*Inferno*, XXVI, 106; Alighieri, 2005; 1977).

These comments do not contradict the hypothesis concerning the position of the island of Mount Purgatory.

6 ULYSSES' FINAL VOYAGE IN THE DIVINE COMEDY

Although Dante knew Homer (he is quoted many times in the *Divine Comedy* and is mentioned in the *Limbo*, IV Canto), he would not have read the *Odyssey* in Greek. Nevertheless he knew the story of Ulysses from various Latin sources (e.g. Ovid's *Metamorphosis* (n.d.) and *Odyssey* by Livius Andronicus (284-c.204 BC; 1999)) and many medieval novels. Based on these sources, Dante virtually invents the story of Ulysses' last voyage. The quest for knowledge was the aim of the voyage, as Dante reveals in *The Banquet* "Naturally all the men desire to know ... " (I, i, 1). Ulysses did not know the Graces, so his knowledge was confined to the purely terrestrial realm:

... unto your senses ... (v. 115) [and] ... was
able to defeat me the longing I have to gain
experience of the world and of the vices and
the worth of men. (vv. 97-99).
(*Inferno*, XXVI, 106; Alighieri, 2005; 1977).

Nor does he direct this knowledge towards a useful goal (rather, it remains always an end in itself), and his desires therefore become negative, so much so that he involves his companions in evil (v. 119-120). Ulysses passes the Pillars of Hercules, beyond which men should not venture, and in breaking a divine prohibition he is defeated by God, and " ... the sea closed over us." (*Inferno*, XXVI, v.142; Alighieri, 2005; 1977).

In some commentaries on Dante's *Divine Comedy* the authors suppose that Dante was inspired by the voyage Ugolino and Vadino Vivaldi made at the end of the thirteenth century. Let us now examine this voyage.

6.1 The Voyage of the Vivaldi Brothers

Vadino and Ugolino Vivaldi were two brothers and Genoese explorers and merchants. They were the first to embark on an oceanic voyage in search of a route from Europe to India. The Vivaldi brothers and Tedisio Doria organized an expedition with two galleys, which left Genoa in May 1291 under the command of Ugolino Vivaldi, accompanied by his brother and two Franciscan friars. Their objective was to reach India by sea, and return to Italy with useful items of trade. The two well-armed galleys sailed down the Moroccan coast to a place called Gozora (Cape Nun), at a latitude of 28°47' N, after which nothing more was ever heard of them (Giano li D'Aragona, 1968).

In 1315, Sorleone de Vivaldo (Ugolino's son) began a series of distant 'wanderings' in an unsuccessful search for his father, and it is said that he even reached Mogadishu on the Somali coast (*ibid.*).

In 1455 another Genoese seaman, Antoniotto Usodimare (1416-1461), sailing with Alvise Da Mosto detto Cadamosto or Ca 'da Mosto (1430-1483) in the service of Prince Henry the Navigator (1394-1460; Major, 1868) of Portugal, claimed to have met near the mouth of the Gambia River the last descendant of the survivors of the Vivaldi expedition. A letter by Usodimare reporting on this voyage is preserved in the University of Genoa Library (see Gråberg, 1802) and relates that the two galleys had sailed into the Sea of Guinea where one was stranded, but the other vessel continued and reached a place on the coast of Ethiopia-Mena or Amenuan, near the 'Gihon' (probably the Senegal River) where the Genoese were conducted to Priest Gianni who documented the story through a letter to the Emperor of Byzantium.

Jacopo Doria Cadamosto (1233-1290), the Genoese chronicler and contemporary of Vivaldi's brothers, wrote that the two galleys passed the Canary Islands. Alvise Ca 'da Mosto, the Venetian navigator who was on the same vessel as Antoniotto Usodimare, reported meeting a white man in a village situated between Gambia and Senegal who claimed to be the last descendant of those on the Vivaldi expedition (Prosperi, 2009).

6.2 Homeric Ships

Homeric ships are represented on Greek Geometric Pottery (e.g. see Figure 3). They had a sharp black hull, but as yet were not provided with a ram. The sides were held together by the thwarts, which formed the seats for the oarsmen. At the bow was a raised platform or deck, on which stood the fighting men of the ship; and there was a similar deck at the stern, on which the arms were kept, and under which there was room for stowage. The length of a fifty-oared galley is calculated to have been between 90 and 100 ft (27.4-30.5 m) from stem to stern, with



Figure 3: An example of an early Greek Geometric skyphos showing a Homeric ship

a breadth amidships of 10-12 ft (3.05-3.65 m). The galley was propelled by oars and sails, the mast being raised or lowered as required. The ship was steered by paddles, and the master of the vessel had his place on the forward deck (Coates, 1989).

The mean speed of these galleys is estimated at 5-6 knots (~1.85 km/hr) when they were going downwind, and up to 10 or 11 knots during storms (*ibid.*).

During the night, the relative positions of the stars in different constellations were used for navigation. Associated with these constellations were mythical figures. Thus, in Homeric times navigators knew the Pleiades, Hyades, Orion, Ursa Major, Bootes, Arcturus and the brightest star in their sky, Sirius (see Theodossiou et al., 2011). They also knew that from the Mediterranean area Ursa Major was a circumpolar constellation.

In the first millennium BC the northern point of reference was the star Kochab in the constellation of Ursa Minor, also known as 'Stella Phoenicia', which was $\sim 7^\circ$ from the North Celestial Pole (Facciolo, 2010; Gianoli D'Aragona, 1968).

7 THE HEIGHT AND SURFACE OF MOUNT PURGATORY ACCORDING TO RODOLFO BENINI

As in Section 4, according to Rodolfo Benini Mount Purgatory is found opposite Zion, that is at a latitude of around $23^\circ 30' S$.

Let us now investigate the height and surface of Mount Purgatory.

7.1 The Visible Horizon

Since the lookout on an Homeric ship was located on the bridge and the ship did not have a particularly deep draught, we can suppose that his eyes were around five braccia¹⁰ above mean sea level.

From this height we can determine the linear distance to the horizon as follows. In reference

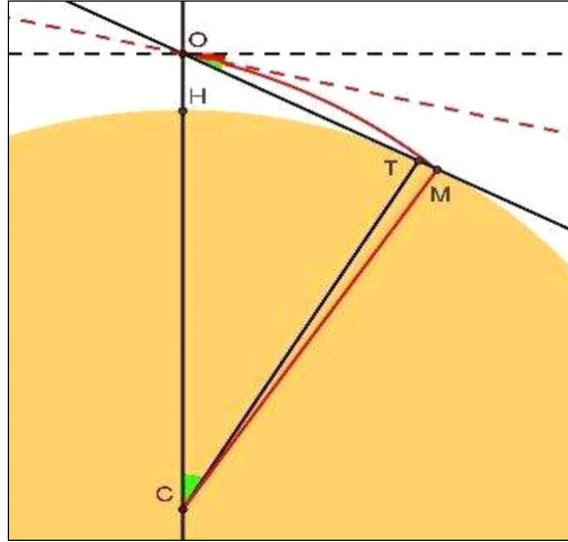


Figure 4: Calculation of the linear distance to the horizon.

to Figure 4, let O represent the eyes of the lookout, H-C the Earth's radius and OT the tangent line to the Earth's surface. The d_T (in radians) of the angle OCT ⁴¹ is given by

$$d_T = \sqrt{[(2 \times OH) / R]} = \sqrt{(2h/R)} \quad (3)$$

Since in this case $OH = 3$ m and $R = 5647$ km (Ptolemy's measure), we have $d_T = 0.00103$, or about 3.5 minutes. Using the formula

$$HT = \sqrt{(2 \times h \times R)} \quad (4)$$

we calculate $HT = 5821$ m, or about 3 nautical miles. Each minute is equivalent to 0.887 nautical miles on the Earth's surface. If we want to consider atmospheric refraction, it is enough to replace R by $1.168R$ in the above formulae.

We obtain for the angle OCM the measure $d_M = 0.000954$, that is 3.28 minutes, and $HM = 6287$ m, or about 3.4 nautical miles. Each minute is equal to about 1 nautical mile on the surface of the Earth.

7.2 The Height and Surface of Mount Purgatory

In reference to Figure 5, we can calculate the height of Mount Purgatory if we can measure the angle $d = FCT$. For this,

$$MF = (Rd^2)/2 \quad (5)$$

We can determine the distance HF on the surface of the Earth between the ship and Mount Purgatory through the following formula:

$$HF = HT + TF = \sqrt{(2 \times UH \times R)} + \sqrt{(2 \times MF \times R)} \quad (6)$$

from which

$$HF = HT + TF = \sqrt{(2 \times UH \times R)} + R \times d \quad (7)$$

In this case, $d = 900 - 3.5 = 896.5$ minutes and $R = 5647.083$ km (Ptolemy's measure), so we obtain $MF = 192$ km = 110 miles.

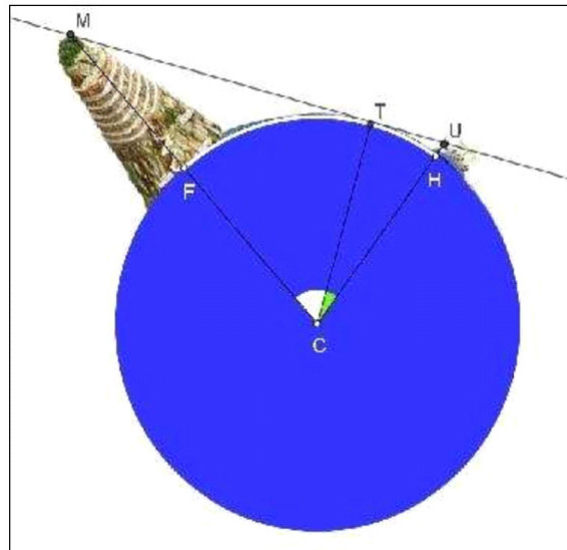


Figure 5: Determination of the height of Mount Purgatory.

According to Benini (1917), who studied Dante's walk after conversing with Manfredi (*Purgatory*, IV), Mount Purgatory is a *perfect cone*, that is, the diameter of the basis is equal to its height (see Table 3). For a comparison of Mount Purgatory's size and two other well known and similarly-sized islands see Table 4.

Table 3: Mount Purgatory's measurements according to Benini (1917).

Height	$h = 192 \text{ km}$
Base diameter	$d = 192 \text{ km}$
Circumference of the base	$C = \pi d = 603 \text{ km}$
Island Area	$A = \pi d^2/4 = 28953 \text{ km}^2$
Volume of Purgatory	$V = A h/3 = 1852987 \text{ km}^3$

Table 4: The sizes of some comparable islands.

Island	Size (km ²)
Purgatory Island	28,953
Timor	28,418
Sicily	25,662

8 THE LOCATION OF MOUNT PURGATORY THROUGH THE OBSERVATION OF THE SOUTHERN CROSS

I turned to the right and gave heed
to the other pole, and saw four stars never seen
before save by the first people.
the heavens seemed to rejoice in their
flames. O northern widowed clime,
that are deprived of beholding them!
(*Purgatory*, I, 22-27; Alighieri, 2005; 1977).

The meaning that is usually attributed to the four stars is that they would be an allegorical representation of the four cardinal virtues: courage, temperance, wisdom and justice. In Dante's vision, the four stars are always visible from the summit of the Mount Purgatory, which is placed on the Earthly Paradise. The commentators do not deny that they may also be stars, that is, the four stars of the Southern Cross.

8.1 The Southern Cross

The constellation of Crux, the Southern Cross, is one of the most conspicuous features of the southern sky, and its principal star, Acrux (α Crucis) is the thirteenth brightest star in the sky. At the present time, the entire constellation is visible from latitude 27° north, while in the Southern Hemisphere it is circumpolar from temperate regions. Where it is visible every night of the year it can be used to locate south: by drawing a straight line from α Crucis to γ Crucis (Gacrux) and extending it approximately 4.5 times the distance between these two stars you reach a point that is very close to the South Celestial Pole, and by dropping a perpendicular from this point to the horizon you get due south. Crux lies on the southern part of the Milky Way, and its brightest stars (with the exception of γ Crucis) are part of the stellar association known as the Crux-Centaurus Association.

The first European to publish a description of this constellation in modern times was the Italian explorer Andrea Corsali (1487-1545), who described it as " ... so fayre and bewtiful, that none other hevenly signe may be compared to it ... " (Corsali, 1516).

The first European cartographer who included the Southern Cross on a celestial atlas was Petrus Plancius in 1598.

On the other hand, the Southern Cross was known to the ancient astronomers, but as part of the constellation of Centaurus (which surrounded it on three sides while the fourth side 'bordered' the constellation of Musca).

Some early European sailors who sailed to the southern hemisphere described the Southern Cross not as a 'cross' but as an 'almond' (e.g. see Vespucci, 1499).

However, due to the precession of the equinoxes (first described by Hipparchus of Nicea in 130 B.C.), the Southern Cross was observable in the Middle Ages by Venetian and Genoese merchants who frequented the port of Alexandria and sailed up the Nile for trade (Facciolo, 2010). It certainly was seen by Knights who participated in the Fifth Crusade (1217-1221) to Egypt under Duke Leopold of Austria, and later by those who took part in the Sixth Crusade (1248-1254) under King Louis IX of France, also to Egypt (*ibid.*). Dante's teacher, Brunetto Latini, mentioned in the *Divine Comedy (Inferno, XV)*, travelled to Spain and probably had a chance to see celestial maps drawn by Arab cartographers that included southern constellations. The Arabs were familiar with a large part of Africa and the Indian Ocean, as shown by a map dating to 1154 (Conder, 1881). During Dante's time, Crux was visible and known to Arab caravaners of the desert, and Arab voyages to the Indian Ocean were described by Rahmanj in 1184, as reported by the fifteenth century astronomer and navigator Ahmad ibn Majid (Facciolo, 2010).

In *the Almagest*, the Southern Cross is not listed but its four main stars all appear as belonging to Centaurus. The Southern Cross could be identified in the constellation referred to as the throne of Caesar, which Plinius (77) described as no longer visible from Italy, but still visible from Egypt (*Natural History, II: 68*). The Southern Cross received its name at the time of the Emperor Augustus (Magli, 2009).

8.2 The Position of the Southern Cross in the Night Sky at the Time of Dante

If we know the declination (δ) of a celestial object and the latitude (ϕ) of the observer's location, we can calculate the maximum and minimum heights of that object relative to the local horizon. Using the derived co-ordinates of the four main stars of the Southern Cross in the fourteenth century, as listed in Table 5, we can calculate the maximum and minimum elevation (h) of Crux at the time of Dante (see Table 6). We find that this constellation was fully visible throughout the night at latitudes greater than or equal to $59^{\circ}12'02''$ S. Therefore, Mount Purgatory must have been located at a latitude of at least $30^{\circ}47'58''$ S.

Table 5: The coordinates of the Southern Cross in the fourteenth century (after Facciolo, 2010).

Star	Right Ascension h:m:s	Declination degree (S)
α Crucis	11h 49m 48s	$59^{\circ} 12' 02''$
β Crucis	12h 08m 54s	$55^{\circ} 49' 02''$
γ Crucis	11h 54m 06s	$53^{\circ} 10' 10''$
δ Crucis	11h 39m 38s	$54^{\circ} 50' 53''$

Table 6: Maximum and minimum height of the Southern Cross. *

Star	h maximum	h minimum
α Crucis	$149^{\circ} 12' 02'' + \phi $	$-30^{\circ} 47' 58'' + \phi $
β Crucis	$145^{\circ} 49' 02'' + \phi $	$-34^{\circ} 10' 58'' + \phi $
γ Crucis	$143^{\circ} 10' 10'' + \phi $	$-36^{\circ} 49' 50'' + \phi $
δ Crucis	$144^{\circ} 50' 53'' + \phi $	$-35^{\circ} 09' 07'' + \phi $

* In this table we have ignored the effect of stellar extinction near the horizon, which is anyhow small for first magnitude stars. For instance, according to Thom's rule, a first magnitude star is first visible at an altitude of 1° .

9 THE HEIGHT AND SURFACE OF MOUNT PURGATORY THROUGH OBSERVATION OF THE SOUTHERN CROSS

Now we can proceed to estimate the height of Mount Purgatory by calculating the apparent depression (dep), as given in the thesis by Capasso (1967). For $\Delta T=15$, $\phi=30^{\circ}30'$, $\delta=6^{\circ}$; there is a measure of $dep = 192'$. Since the elevation (e) of the human eye is estimated as $e = (dep/1.8)^2$, we get $e = 11,429$ meters. Since the entrance to the Garden of Eden is near the top of Mount Purgatory, the mountain cannot have been much higher than this.

We calculate the surface area and volume of Purgatory Island, using the average slope of Capasso (Table 7). This indicates that Purgatory Island had an area comparable to that of the island of Lolland in Denmark (1243 km^2).

Now we calculate the surface area and volume of Purgatory Island, following the direct calculations of Benini (see Table 8). In this instance, Purgatory Island had an area that was comparable to that of the island of Sant'Antioco in Italy (115.6 km^2).

Table 7: New measures according to Capasso.

Height	$h=11.4 \text{ km}$
Base diameter	$d=40 \text{ km}$
Circumference of the base	$C = \pi d = 125 \text{ km}$
Island Area	$A = \pi d^2/4 = 1257 \text{ km}^2$
Volume of Purgatory	$V = A h/3 = 4775 \text{ km}^3$

Table 8: New measures according Benini.

Height	$h=11.4 \text{ km}$
Base diameter	$d=11.4 \text{ km}$
Circumference of the base	$C = \pi d = 35.8 \text{ km}$
Island Area	$A = \pi d^2/4 = 102 \text{ km}^2$
Volume of Purgatory	$V = Ah/3=388 \text{ km}^3$

10 CONCLUSION

The foregoing analysis indicates that Mount Purgatory was 11.4 km high and was structured like a ziggurat.

The most notable prehistoric ziggurat was the Etemenanki Temple of the 'foundation of heaven and earth', dedicated to Marduk, in the city of Babylon, which dates to the sixth century BC and the Neo-Babylonian Dynasty. This temple is described on a cuneiform tablet from Uruk dating to 229 BC and now in the Louvre in Paris, which is a copy of an older text. This tablet gives the height of the tower as seven stocks (i.e. 91 meters) with a square base of 91 meters on each side. The existence of this mud brick structure was confirmed by excavations conducted by Robert Koldewey from 1913 onwards (see George, 2005/2006; Schmidt, 1995). Large stairs were discovered on the southern side of the building, where a triple gate connected it with the Esagila, a temple dedicated to Marduk, the protector god of Babylon. A larger gate on the eastern side connected the temple with a road that was used for sacred processions.

It would seem that Dante had in mind a Mount Purgatory that was exactly this shape and contained seven terraces, just like the Etemenanki Temple.

11 NOTES

1. Italian unification (Italian Risorgimento, meaning 'the Resurgence') was the political and social movement in the nineteenth century that consolidated different states on the Italian peninsula into the single state of the Kingdom of Italy. The process began in 1815 with the Congress of Vienna and ended in 1871 when Rome became the capital of the Kingdom of Italy.
2. Dante's verses cited in this paper were translated into English by Charles Southward Singleton (see Alighieri, 1977). For another important English prose translation, by Dorothy L. Sayers, see Alighieri (1955).
3. We presume that this relates to the Florentine mile, which was used at the time of Dante (Benivieni, 1897).
4. Note that this mathematical use of ΔT differs from its conventional use in astronomy and Earth rotation studies.
5. Three times the average height of a typical human body would be about five meters.
6. Following the tradition established by the Fathers and the Scholastics, Dante attaches an immeasurable height to the Mount.
7. The Wain' were the seven brightest stars in the constellation of Ursa Major.
8. The equinoctial colure is the meridian or great circle of the celestial sphere which passes through the celestial poles and the two equinoxes.
9. The great vertical is a great circle of the celestial sphere whose plane is perpendicular to that of the horizon.
10. The 'Fiorentine braccio' (plural: braccia) is about 0.58 m.
11. $OC \times \cos(d_T) = CT \rightarrow \cos(d_T) = R/(R + h) \rightarrow 1 - (d_T^2)/2 + O(1) = 1 - h/R + O(1)$

12 ACKNOWLEDGEMENTS

The authors thank Professor Corradino Astengo (Università di Genova) and Dr Franca Acerenza, (Istituzione Musei del Mare e della Navigazione, Genova) for useful explanations of ancient cartography, and Professor Giulio Magli, (Politecnico di Milano) for encouragement.

13 REFERENCES

- Alfraganus, M., 833. *Elements of Astronomy*.
- Alhazen, A., 2007. Alhacen's theory of visual perception. *Transactions of the American Philosophical Society*, 91 (4), 1-550.
- Alhazen, A., 1972 *Theory of Visual Perception*. New York, Johnson Reprint Corporation. Pp. 283-288.
- Alighieri, D., 1955. *The Divine Comedy. Part 2: Purgatory*. Harmondsworth, Penguin Books (translated with a commentary by D.L. Sayers).
- Alighieri, D., 1977. *The Divine Comedy. Inferno, Purgatory, Paradise*. Princeton, Princeton University Press (translated with a commentary by C.S. Singleton).
- Alighieri, D., 2005. *The Divine Comedy*. Milano, Mondadori (in Italian).
- Andronicus, L., 1999. *Odyssey* (English translation by David Camden).
- Angelini, A., Magnaghi-Delfino, P., Norando, T., 2014.
- Galileo Galilei's location, shape and size of Dante's Inferno: an artistic and educational project. In *Proceedings 13th Conference on Applied Mathematics APLIMA T*. Bratislava, Institute of Mathematics and Physics, Sjf STU. Pp. 7-20.
- Angelitti, F., 1897. On the date of Dante's voyage derived from historical data and confirmed by astronomical observations reported in the Comedy. *Atti dell'Accademia Pontaniana*, XXVII, 1-114 (in Italian).
- Aristotle, 1931. *Meteorology*. In Webster, E.W. (English translation). *The Works of Aristotle*. Oxford, Clarendon Press Pp.1-484.
- Bacon, R., 1962. *The Opus Majus of Roger Bacon. Volume II* (English translation by R.B. Burke). Ressel & Rullell Inc.
- Benini, R., 1917. Origin, site, size and shape of Mount Purgatory and of Dante's Inferno. *Rendiconti della reale Accademia dei Lincei, Classe di Scienze Morali, Storiche e Filologiche*, 25(11), 1-117 (in Italian).
- Benivieni, G., 1897. *Dialogue of Antonio Manetti Florentine Citizen about the Size, the Form and the Measurements of the Inferno of Dante Alighieri a cura di N. Zingarelli*. Città di Castello, Lapi (in Italian).
- Bonatti, G., 1851. Book of Astronomy. In Boncampagni, B. *On the Life and Works of Guido Bonatti, Astrologer and Astronomer of the Thirteenth Century*. Rome, Tipografia delle Belle Arti.
- Cantelli, F.P., 1917. The knowledge of the times in Dante's journey. *Tipografia Matematica*, 36, 1-10 (in Italian).
- Capasso, I., 1967. *Astronomy of the Divine Comedy*. Pisa, Domus Galileiana (in Italian).
- Coates, J.F., 1989. The trireme sails again. *Scientific American*, 261 (4), 68-75.
- Conder, C.R., 1881. *Mediaeval Topography of Palestine*. London, Charles Wilson.
- Corsali, A., 1516. *The Letter of Andrea Corsali to the illustrious Prince Duke de Medici*. Translated into English by Richard Eden in 1555.
- Facciolo, C., 2010. *The Long Journey of the Southern Cross*. Verona, Edizioni Il Frangente (in Italian).
- Ferrara, O., 1998. *The Lords of the Sea*. Sarno, Centro Studi I Dioscuri (in Italian).
- Galilei, G., 1588. Two lectures to the Florentine Academy on the shape, location and size of Dante's Inferno. Translated into English by Mark A. Peterson.
- George, A.R., 2005/2006. The Tower of Babel: archaeology, history and cuneiform texts. *Archiv fur Orientforschung*, 51,75-95.

- Gianoli D'Aragona, M., 1968. *Seafaring Adventures*. Pisa, Nitri-Lischi (in Italian).
- Gräberq, G., 1802. *Genova. Two Volumes*. Genova, Annali di Geografia e di Statistica (in Italian).
- Janni, P., 1996. *The Sea of the Ancients*. Bari, Edizioni Dedalo (in Italian).
- Kretschmer, K., 1891. Marin Sanudo the Elder and the maps of Peter Visconte. *Zeitschrift der Gesellschaft für Erdkunde zu Berlin* (in German).
- Kretschmer, K., 1909. *The Italian Pilots' Book of the Middle Ages*. Berlin, Institut und Museum für Meereskunde (in German).
- Latini, B., n.d. *The Tresor*.
- Magli, G., 2009. *The Time of the Cyclops*. Bologna, Pitagora (in Italian).
- Major, R.H., 1868. *Life of Prince Henry of Portugal*.
- London, Asher & Co.
- Ovid, n.d. *Metamorphosis*.
- Plancius, P., 1598. Star maps.
- Plinius, G., 77. *Natural History*. Translated into English by John Bostock in 1855.
- Posidonius, n.d. . Cited in Cleomedes, 2000. On the circular motions of the celestial bodies. *Perception*, 29, 863-871.
- Prospero, F., 2009. *Two Sails far a Dream*. Milano, Mursia (in Italian).
- Reynolds, M., 1999. A comparative geometric analysis of the heights and bases of the Great Pyramid of Khufu and the Piramide de Sole at Teotihuacan. *Nexus Network Journal*, 1, 23-42.
- Sanudo, M., 1972. *The Book of the Secrets of the Faithful of the Cross*. Toronto, University of Toronto (in Latin).
- Schmidt, H., 1995. *The Etemenanki Tempie in Babylon*. Mainz am Rein, Philipp von Zabern (in German).
- Theodossiou, E., Manimanis, V.N., Mantarakis, P., and Dimitrijevic, M.S., 2011. Astronomy and constellations in the *Iliad* and *Odyssey*. *Journal of Astronomical History and Heritage*, 14,22-30.
- Vespucci, A., 1499. Drawing of the Southern Cross.